Module-5

TORSION OF CIRCULAR SHAFTS AND ELASTIC STABILITY OF COLUMNS

Syllabus

Torsion:

Introduction, Pure torsion, assumptions, derivation of torsional equations, polar modulus, torsional rigidity / stiffness of shafts, Power transmitted by solid and hollow circular shafts

TORSION OF CIRCULAR SHAFTS

INTRODUCTION

In this chapter structural members and machine parts that are in *torsion* will be considered. More specifically, the stresses and strains in members of circular cross section subjected to twisting couples, or *torques,* T and T' (Fig. 5.1) are analyzed. These couples have a common magnitude *T,* and opposite senses. They are vector quantities and can be represented by couple vectors as shown in Fig.5.1.

Members in torsion are encountered in many engineering applications. The most common application is provided by *transmission shafts,* which are used to transmit power from one point to another. For example, the shaft shown in Fig. 5.1 is used to transmit power from the engine to the rear wheels of an automobile. These shafts can be solid, as shown in Fig. 5.1, or hollow.

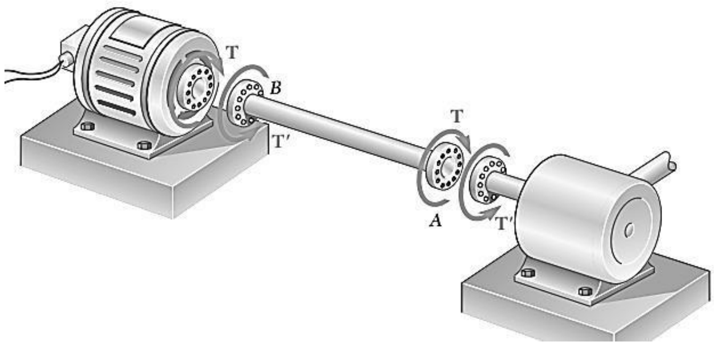

Fig. 5.1: Torsion in shafts

PURE TORSION

A member is said to be in pure torsion when its cross sections are subjected to only torsional moments and not accompanied by axial forces or bending moment. Now consider the section of a shaft under pure torsion as shown in Fig. 5.2.

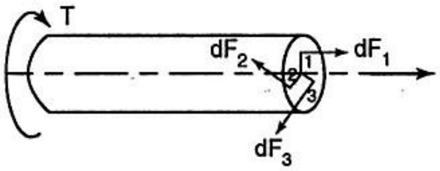

Fig. 5.2 Pure torsion

The internal forces develop so as to counteract this torsional moment. Hence, at any element, the force *dF* developed is in the direction normal to radial direction. This force is obviously shearing force and thus the elements are in pure shear. If *dA* is the area of the element at distance *r* from the axis of shaft, then,

$$dF = \tau\, dA$$

where τ is shearing stress,

and $$dT = dF \times r$$

ASSUMPTIONS IN THE THEORY OF PURE TORSION

In the theory of pure torsion, expressions will be derived for determining shear stress and the effect of torsional moment on cross-section *i.e.* in finding angle of twist. In developing this theory the following assumptions are made.

- The material is homogeneous and isotropic.
- The stresses are within the elastic limit, *i.e.* shear stress is proportional to shear strain.
- Cross-sections which are plane before applying twisting moment remain plane even after the application of twisting moment *i.e.* no warping takes place.
- Radial lines remain radial even after applying torsional moment.
- The twist along the shaft is uniform.

DERIVATION OF TORSIONAL EQUATIONS

Consider a shaft of length L, radius *R* fixed at one end and subjected to a torque Tat the other end as shown in Fig. 5.3.

Let O be the centre of circular section and B a point on surface. AB be the line on the shaft parallel to the axis of shaft. Due to torque T applied, let *B* move to *B'*. If γ is shear strain *(angle BOB')* and θ is the angle of twist in length *L*, then

$$R\theta = BB' = L\gamma$$

If τ_s is the shear stress and G is modulus of rigidity then,

$$\gamma = \frac{\tau}{G}$$

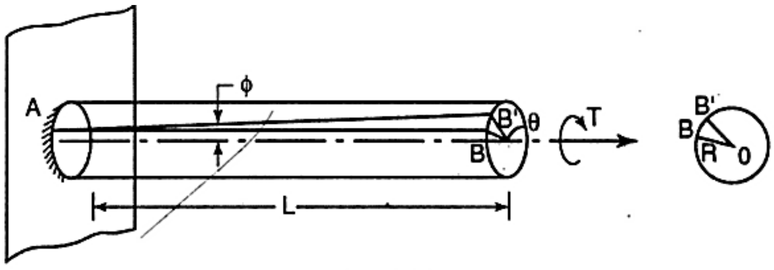

Fig. 5.3: Torsion in shaft

$$R\theta = L\frac{\tau_s}{G}$$

$$\frac{\tau_s}{R} = \frac{G\theta}{L}$$

Similarly if the point *B* considered is at any distance *r* from centre instead of on the surface, it can be shown that

$$\frac{\tau}{r} = \frac{G\theta}{L} \qquad \ldots \text{(i)}$$

$$\frac{\tau_s}{R} = \frac{\tau}{r}$$

Thus shear stress increases linearly from zero at axis to the maximum value τ_s at surface.

Now consider the torsional resistance developed by an elemental area 'δa' at distance *r* from centre.

If τ is the shear stress developed in the element the resisting force is

$$dF = \tau da$$

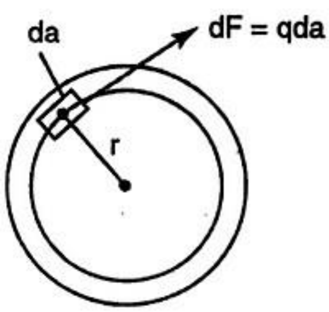

Fig. 8.4

Resisting torsional moment,

$$dT = dF \times r$$
$$= \tau r da$$

WKT,

$$\tau = \tau_s \frac{r}{R}$$

Therefore,

Total resisting torsional moment, $$dT = \tau_s \frac{r^2}{R} da$$

$$T = \sum \tau_s \frac{r^2}{R} da$$

$$T = \frac{\tau_s}{R} \sum r^2 da$$

But $\sum r^2 da$ is nothing but polar moment of inertia of the section. Representing it by notation J we get,

$$T = \frac{\tau_s}{R} J$$

i.e., $$\frac{T}{J} = \frac{\tau_s}{R}$$

WKT, $$\frac{\tau_s}{R} = \frac{\tau}{r}$$

There, $$\frac{T}{J} = \frac{\tau}{r} \qquad \text{(ii)}$$

From (i) and (ii), we have,

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L} \qquad \text{(iii)}$$

Where,

T - torsional moment , N-mm

J - polar moment of inertia, mm^4

τ - shear stress in the element, N/mm^2

r- distance of element from centre of shaft, mm

G - modulus of rigidity, N/mm^2

θ- angle of twist, rad

L- length of shaft, mm

POLAR MODULUS

From the torsion equation,

$$\frac{T}{J} = \frac{\tau}{r}$$

But,

$$\frac{\tau_s}{R} = \frac{\tau}{r}$$

Where τ_s is maximum shear stress (occurring at surface) and R is extreme fibre distance from centre. Therefore,

$$\frac{T}{J} = \frac{\tau_s}{R}$$

or

$$T = \frac{J}{R}\tau_s = Z_p \tau_s$$

where Z_p is called as 'Polar Modulus of Section'. It may be observed that Z_p is the property of the section and may be defined as the ratio of polar moment of inertia to extreme radial distance of the fibre from the centre.

(i) **For solid circular section of diameter *d***

$$J = \frac{\pi}{32} d^4$$

$$R = \frac{d}{2}$$

$$Z_p = \frac{J}{R} = \frac{\pi}{16} d^3$$

$J \quad R$

(ii) **For hollow circular shaft with external diameter d_1 *and* internal diameter d_2**

$$J = \frac{\pi}{32}(d_1^4 - d_2^4)$$

$$R = \frac{d_1}{2}$$

$$Z_p = \frac{J}{R} = \frac{\pi}{16} \frac{d_1^4 - d_2^4}{d_1}$$

TORSIONAL RIGIDITY / STIFFNESS OF SHAFTS

From the torsion equation,

Angle of twist, $$\theta = \frac{TL}{GJ}$$

T - Torsional moment , N-mm

J - Polar moment of inertia, mm^4

G - Modulus of rigidity, N/mm^2 (sometimes denoted by C)

θ - angle of twist, rad

For a given specimen, the shaft properties like length L, polar modulus J and material properties like rigidity modulus G are constants and hence the angle of twist is directly proportional to the twisting moment or torque producing the twist. Torque producing twist in a shaft is similar to the bending moment producing bend or deflection in a beam. Similar to the flexural rigidity in beams expressed by EI, torsional rigidity is expressed as GJ which can be defined as the torque required to produce a twist of unit radian per unit length of the shaft.

POWER TRANSMITTED

Let us consider a circular shaft running at N rpm under mean torque T. Let P be the power transmitted by the shaft in kW.

The angular speed of the shaft is given by the distance covered by a particle in the circle in radians for N revolutions per second, i.e. the particle covers 2π radians for one revolution and for N revolutions the particle covers $2\pi N$ radians in one minute. Hence the angular speed ω is given by:

$$\omega = \frac{2\pi N}{60} \text{ Rad/s}$$

Thus, the power transmitted = Mean torque (kN-m) x Angular speed (rad/s)

i.e.,

$$P = T\omega = \frac{2\pi NT}{60} \text{ kN-m/s or kW}$$

It is seen that from the above equation *mean torque T* in kN-m is obtained. It should be converted to N-mm so that the stress due to torque can be obtained in N/mm^2. Maximum shear stress due to torque can be obtained from the torque equation.

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

WORKED EXAMPLES

1) A solid shaft has to transmit 120 kW of power at 160 rpm. If the shear stress is not to exceed 60 MPa and the twist in a length of 3 m must not exceed 1°, find the suitable diameter of the shaft. Take G = 80 GPa.

Solution

P = 120 kW, N = 160 rpm, τ = 60 N/mm^2, θ = 1°, G or C = 80 x 10^3 N/mm^2, d = ?

Power transmitted is given by,

Power transmitted is given by $P = \frac{2\pi NT}{60}$

$$T = \frac{120 \times 60}{2 \times \pi \times 160} = 7.162 \text{ kN-m} = 7.162 \times 10^6 \text{ N-mm}$$

From torque equation, we have

$$\left[\frac{T}{J} \text{ or } \frac{T}{I_P} = \left[\frac{\tau_s}{R}\right] = \frac{C\theta}{l}\right]$$

$$\text{where } J = \frac{\pi R^4}{2}$$

(i) From the maximum shear stress considerations

$$J = \frac{\pi R^4}{2} = \left[\frac{R}{\tau_s}\right] \times T = \frac{R}{60} \times 7.162 \times 10^6$$

$$R = \left[\frac{2 \times (7.162 \times 10^6)}{60\pi}\right]^{\frac{1}{3}} = 42.357 \text{ mm}$$

(ii) From the maximum twist considerations

$$l = 3 \text{ m}, \ \theta = 1^\circ = 1 \times \frac{\pi}{180} \text{rad}$$

$$\frac{T}{I_P} = \frac{C\theta}{l}$$

$$\frac{\pi R^4}{2} = \frac{7.162 \times 10^6 \times 3000}{(80 \times 10^3)\left(\frac{\pi}{180}\right)} = 55.946 \text{ mm}$$

d = 2 x 55.946 = 111.89 mm

Choose the higher diameter among the two so that it can be safe.

2) Find the diameter of the shaft required to transmit 60 kW at 150 rpm if the maximum torque exceeds 25% of the mean torque for a maximum permissible shear stress of 60 MN/mm^2. Find also the angle of twist for a length of 4 m. Take G = 80 GPa.

Solution

P = 60 kW, N = 150 rpm, τ_s = 60 N/mm^2, θ = ?, G or C = 80 x 10^3 N/mm^2, d = ?

Power transmitted is given by,

$$P = \frac{2\pi NT}{60}$$

$$T = \frac{60 \times 60}{2 \times \pi \times 150} = 3.8197 \text{ kN-m} = 3.8197 \times 10^6 \text{ N-mm}$$

T_{max} = 1.257 = 1.25 x 3.8197 x 106 = 4.77465 x 10^6 N mm.

From torque equation, we have

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{L}$$

Where, $J = \frac{\pi d^4}{32} = \frac{\pi R^3}{2}$

(i) Diameter

$$J = \frac{\pi R^4}{2} = \left[\frac{R}{\tau_s}\right] \times T = \frac{R}{60} \times 4.77465 \times 10^6$$

$$R = \left[\frac{2 \times (4.77465 \times 10^6)}{60\pi}\right]^{\frac{1}{3}} = 37 \text{ mm}$$

(ii) Angle of twist l = 4 m, θ = ?

$$\frac{T}{J} = \frac{C\theta}{l}$$

$$\frac{\pi R^4}{2} = \frac{Tl}{C\theta}$$

$$\theta = \frac{2(4.77465 \times 10^6) \times 4000}{(80 \times 10^3) \times \pi \times 37^4} = 0.0811 \text{ rad} = \frac{180}{\pi} \times 0.0811 = 4.646^\circ.$$

3) A solid cylindrical shaft is to transmit 300 kW power at 100 r.p.m. (a) If the shear stress is not to exceed 80 N/mm^2, find its diameter. (b) What percent saving in weight would be obtained if this shaft is replaced by a hollow one whose internal diameter equals to 0.6 of the external diameter, the length, the material and maximum shear stress being the same?

Solution:

Given:

Power, $P = 300$ kW $= 300 \times 10^3$ W

Speed, $N = 100$rpm

Max. Shear stress, $\sigma = 80$ N/mm^2

(a)

Let D = Dia. of solid shaft

Power transmitted by the shaft is given by,

$$P = \frac{2\pi NT}{60}$$

$$300 \times 10^3 = \frac{2\pi \times 100 \times T}{60}$$

$$T = \frac{300 \times 10^3 \times 60}{2\pi \times 100} = 28647.8 \text{ Nm} = 28647800 \text{ Nmm}$$

$$T = \frac{\pi}{16} \times \tau \times D^3 \quad \text{or} \quad 28647800 = \frac{\pi}{16} \times 80 \times D^3$$

$$D = \left(\frac{16 \times 28647800}{\pi \times 80}\right)^{1/3} = 121.8 \text{ mm}$$

$$= \textbf{say 122.0 mm.} \quad \textbf{Ans.}$$

(b) Percent saving in weight

Let D_0 = External dia. of hollow shaft D_i. = Internal dia. of hollow shaft = $0.6 \times D_o$. (given)

The length, material and maximum shear stress in solid and hollow shafts are given the same.

Hence torque transmitted by solid shaft is equal to the torque transmitted by hollow shaft.

But the torque transmitted by hollow shaft is given by equation,

$$T = \frac{\pi}{16} \times \tau \times \frac{(D_0{}^4 - D_i{}^4)}{D_0}$$

$$= \frac{\pi}{16} \times 800 \times \frac{[D_0{}^4 - (0.6\, D_0)^4]}{D_0}$$

$$= \pi \times 50 \times \frac{[D_0{}^4 - (0.6\, D_0)^4]}{D_0}$$

But torque transmitted by solid shaft = 28647800 N-mm.

Equating the two torques, we get

$$28647800 = \pi \times 50 \times \left(\frac{0.8704\, D_0^4}{D_0}\right) = \pi \times 50 \times 0.8704\, D_0^3$$

$$D_0 = \left(\frac{28647800}{\pi \times 50 \times 0.8704}\right)^{1/3} = 127.6 \text{ mm} = \text{say } 128 \text{ mm}$$

Internal dia, $D_i = 0.6 \text{ x } D_0 = 0.6 \text{ x } 128 = 76.8$ mm

Let, W_s = Weight of solid shaft,

W_h = Weight of hollow shaft.

Let, W_s = Weight density x Area of solid shaft x Length

$$= w \times \frac{\pi}{4} D^2 \times L$$

Similarly,

$$W_h = \text{Weight density} \times \text{Area of hollow shaft} \times \text{Length}$$

$$= w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L$$

Now percent saving in weight

$$= \frac{W_s - W_h}{W_s} \times 100$$

$$= \frac{w \times \frac{\pi}{4} D^2 \times L - w \times \frac{\pi}{4} [D_0^2 - D_i^2] \times L}{w \times \frac{\pi}{4} D^2 \times L} \times 100$$

$$= \frac{D^2 - (D_0^2 - D_i^2)}{D^2} \times 100 \qquad \left(\text{Cancelling } w \times \frac{\pi}{4} \times L\right)$$

$$= \frac{122^2 - (128^2 - 75.8^2)}{122^2} \times 100 = \frac{14884 - (16364 - 5898)}{14884} \times 100$$

$$= \frac{14884 - 10486}{14884} \times 100 = \mathbf{29.55\%}. \quad \textbf{Ans.}$$

4) A hollow shaft of diameter ratio 3/8 is to transmit 375 kW power at 100 r.p.m. The maximum torque being 20% greater than the mean. The shear stress is not to exceed 60 N/mm^2 and twist in a length of 4 m not to exceed 2°. Calculate its external and internal diameters which would satisfy both the above conditions. Assume modulus of rigidity, **C** = 0.85 x **10⁵** N/mm^2.

Solution:

Diameter ratio, $\frac{D_i}{D_0} = \frac{3}{8}$

$\therefore$ $D_i = \frac{3}{8} D_0$

Power, $P = 375 \text{ kW} = 375000 \text{ W}$

Speed, $N = 100$ r.p.m.

Max. torque, $T_{max} = 1.2\, T_{mean}$

Length, $L = 4 \text{ m} = 4000 \text{ mm}$

Max. twist, $\theta = 2° = 2 \times \frac{\pi}{180} \text{ radians} = 0.0349 \text{ radians}$

Modulus of rigidity, $C = 0.85 \times 10^5 \text{ N/mm}^2$

Power is given by, $P = \frac{2\pi NT}{60}$ Here torque is T_{mean}

$$T = \frac{P \times 60}{2\pi N} = \frac{375000 \times 60}{2\pi \times 100} = 35810 \text{ Nm}$$

$$T_{mean} = 35810 \text{ Nm}$$

$$\therefore \quad T_{max} = 1.2 \times T_{mean} = 1.2 \times 35810$$

$$= 42972 \text{ Nm} = 42972 \times 1000 \text{ Nmm.}$$

i) Diameters of the shaft when shear stress is not to exceed 60 MPa,

For the hollow shaft, the torque transmitted is given by

$$T_{max} = \frac{\pi}{16} \times \tau \times \frac{(D_0^4 - D_i^4)}{D_0}$$

$$42972 \times 1000 = \frac{\pi}{16} \times 60 \times \frac{\left[D_0^4 - \left(\frac{3}{8} D_0\right)^4\right]}{D_0}$$

$$\frac{42972000 \times 16}{\pi \times 60} = \frac{D_0^4}{D_0}\left(1 - \frac{81}{4096}\right) = D_0^3 \times \frac{4015}{4096}$$

$$D_0^3 = \frac{42972000 \times 16 \times 4096}{\pi \times 60 \times 4015}$$

$$D_0 = \left(\frac{42972000 \times 16 \times 4096}{\pi \times 60 \times 4015}\right)^{1/3} = 154.97 \text{ mm say } 155 \text{ mm}$$

$$D_i = \frac{3}{8} D_0 = \frac{3}{8} \times 155 \simeq 58.1 \text{ mm}$$

(ii) Diameters of the shaft when the twist is not to exceed 2 degrees.

$$\frac{T}{J} = \frac{C \times \theta}{L}$$

$$\frac{42972000}{\frac{\pi}{32}[D_0^4 - D_i^4]} = \frac{(0.85 \times 10^5) \times 0.0349}{4000}$$

$$\frac{42972000 \times 4000 \times 32}{\pi \times 0.85 \times 10^5 \times 0.0349} = D_0^4 - D_i^4 = D_0^4 - \left(\frac{3}{8} D_0\right)^4 = D_0^4 - \frac{81}{4096} D_0^4$$

$$= D_0^4 \left[1 - \frac{81}{4096}\right] = \frac{4015}{4096} D_0^4$$

$$\therefore \quad D_0^4 = \frac{42972000 \times 4000 \times 32 \times 4096}{\pi \times 0.85 \times 10^5 \times 0.0349 \times 4015}$$

$$\therefore \quad D_0 = 156.65 \text{ mm say } 157 \text{ mm}$$

$$D_i = \frac{3}{8} \times 156.65 = 58.74 \text{ mm say } 59 \text{ mm.}$$

The diameters of the shaft, which would satisfy both the conditions, are the greater of the two values.

External dia., D_0 = 157 mm.

Internal dia., D_i = 59 mm.